Critical Mass Thresholds for Neutron Star Stability and Black Hole Formation in Gravitational Collapse

Bijan Kumar Gangopadhyay

Independent Researcher, Chowdhuripara, P.O Makardaha, Dt Howrah,West Bengal 711409, India.

bkgangopadhyay@gmail.com

Abstract

This research investigates the critical mass thresholds for black hole formation during the gravitational collapse of massive stars. Using numerical simulations and analytical techniques, we model the collapse of spherically symmetric, non-rotating neutron stars by solving the Tolman-Oppenheimer-Volkoff (TOV) equation. We first derive an analytical solution for the TOV equation under the assumption of constant density, estimating the maximum neutron star mass to be 2.85 solar masses. We then incorporate a customized density profile, as predicted in our previous work, into the TOV framework. This yields a critical mass of 2.096 solar masses at a radius of approximately 10 km, consistent with current theoretical and observational expectations. The maximum stable mass with this profile is calculated to be 2.36 solar masses, with the mass decreasing to zero beyond 15.5 km. By analyzing different initial masses (2.0, 4.0, and 8.0 solar masses) using a polytropic equation of state (EOS), we examine the mass-radius and pressure-radius relationships. Our results reveal a highly non-linear and abrupt change in mass and pressure distributions, indicating the formation of a dense outer shell. This structural feature could significantly influence neutron star stability and the conditions leading to black hole formation. These findings provide valuable insights into the maximum mass limits of neutron stars, aiding in the interpretation of astrophysical observations and the identification of potential black hole progenitors.

Keywords: Neutron Star, Tolman-Oppenheimer-Volkoff Equation, Black Hole, Gravitational Collapse, Stellar Stability.
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*** Address of correspondence**
Bijan Kumar Gangopadhyay
Independent Researcher, Chowdhuripara, P.O Makardaha, Dt Howrah,West Bengal 711409, India.
Email: bkgangopadhyay@gmail.com

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Introduction

The probability of existence of neutron star was predicted by scientists after the discovery of neutron by Chadwick in 1932. Among them pioneers were Lev Landau, Walter Baade and Fritz Zwicky (1936). They believed that during supernova explosion of a main sequence star of more than 10 solar mass, neutron star might be formed. However, the history of observation of neutron star begins in 1967 when Jocelyn Bell and her advisor Anthony Hewish discovered radio pulsar (PSR B1919+21) at Cambridge. Various observation and theoretical prescriptions now lead us to believe that pulsars are actually rapid rotating neutron stars. It is difficult to detect any non-rotating neutron star. So, analyzing the radiation emitted by pulsars, properties of neutron stars can be predicted. In 1939 Oppenheimer and E. Salpeter along with different theoreticians realized that like white dwarf, neutron star might have an upper mass limit. Oppenheimer and Volkoff derived an equation of hydrostatic equilibrium of star by using general theory of relativity, as they believed that general theory of relativity might be more fruitful than Newtonian mechanics. They calculated the limiting mass of a neutron star to be 0.7 solar mass [1]. This result seemed to be very low, as it is very much expected that the maximum mass of a neutron star should at least exceed the Chandrasekhar mass limit of 1.4 solar mass [2]. The strong repulsive nuclear force acting between neutrons probably increases the upper mass limit of a neutron star. Different stellar observation of pulsars leads to an idea, that maximum mass of neutron star may lie between 1.4 to 3.0 solar mass. Integrating equation of equilibrium, Rhoades and Ruffini [3] found that the maximum mass of neutron star to be 3.2 solar mass. Nauenberg and Chapline (1973) [4] found this to be 3.6 solar mass. Rotating neutron stars may have higher mass limit. Hartle and Sabbadini (1977) [5] introduced an empirical relation for non-rotating neutron star as,

$$M_{\text{bound}} = 11.4 M_{\odot} \left(\frac{10^{17}}{\rho_0}\right)^{\frac{1}{2}} \tag{1}$$

Friedman and Ipser (1987) [6] have derived an empirical relation for rotational neutron star as,

$$M_{\text{MAX}}^{\text{ROT}} = 14.3M_{\odot}\left(\frac{10^{17}}{\rho_0}\right)^{\frac{1}{2}} \quad (2)$$

If we use polytrope stellar model with n=1.5, numerically a standard relation [7] can be employed for evaluating mass of neutron star from MR^3 = constant as,

$$M = 1.102\left(\frac{\rho_0}{10^{18}}\right)^{\frac{1}{2}} M_{\odot} = \left(\frac{15.12Km}{R}\right)^3 M_{\odot} \quad (3)$$

$$R = 14.64\left(\frac{\rho_0}{10^{18}}\right)^{-\frac{1}{6}} \quad (4)$$

In equations (1), (2), (3), and, (4) ρ_0 (density) and R (radius) of neutron star are measured in Kg/m^3 and in Km respectively. Whatever be the nature of equation of state, the maximum mass of a neutron star must be greater than 1.4 solar mass [8]. It is expected that at the end point of stellar evolution, if the core of the star contains carbon or heavier element then there is every possibility that the remnant star might be transformed to neutron star. To retain stability of neutron star, it has a critical mass threshold, beyond which it would be collapsed to a black hole.

Methodology

In this work we have used the following equations as per requirement. Equation of Hydrostatic Equilibrium [9]:

$$\frac{dp(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \quad (5)$$

TOV Equation [10]:

$$\frac{dp(r)}{dr} = -\frac{G\left[m(r) + 4\pi r^3 \frac{p(r)}{c^2}\right]\left[\rho(r) + \frac{p(r)}{c^2}\right]}{r^2\left[1 - \frac{2Gm(r)}{c^2 r}\right]} \quad (6)$$

Mass profile of star (Equation of Continuity) [9]:

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \quad (7)$$

Customized Density profile [11]:

$$\rho(r) = \rho_0\left(1 - \frac{r^2}{R^2}\right) \quad (8)$$

Non-relativistic degeneracy pressure of neutron star [10]

$$P = 0.542 \times 10^4 \rho_0^{5/3} \quad \text{(S. I. Unit)} \quad (9)$$

Relativistic degeneracy pressure of neutron star [10]

$$P = 1.235 \times 10^{10} \rho_0^{4/3} \quad \text{(S. I. Unit)} \quad (10)$$

Analytical Solution of TOV Equation

Let us start with the TOV equation as,

$$\frac{dp}{dr} = -\frac{G}{r^2}\left[\rho + \frac{p}{c^2}\right]\left[m(r) + \frac{4\pi r^3 p}{c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1} \quad (11)$$

As we do not have any authentic equation of state of neutron star, it is almost impossible to solve this equation analytically. So, to solve it analytically, we have assumed here that the density of the relevant star is constant (ρ = constant). As density is constant, we can write, $m(r) = (4/3)\pi r^3 \rho$. So, TOV equation becomes,

$$\frac{dp}{dr} = -\frac{Gm(r)\rho}{r^2}\left[1 + \frac{p}{\rho c^2}\right]\left[1 + \frac{3p}{\rho c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1} \quad (12)$$

Let, $\frac{p}{\rho c^2} = x$, and $\frac{2Gm(r)}{c^2 r} = y$, then we have,

$$\frac{dx}{dr} = -\frac{y}{2r}(1+x)(1+3x)(1-y)^{-1} \quad (13)$$

Now, from $y = \frac{2Gm(r)}{c^2 r}$, it follows:

$$\frac{dy}{dr} = \frac{8\pi G\rho}{3c^2}2r = \frac{y}{r^2}2r = \frac{2y}{r} \quad (14)$$

Now,

$$\frac{dx}{dr} = \frac{dx}{dy}\frac{dy}{dr} = \frac{dx}{dy}\frac{2y}{r} \quad (15)$$

Hence TOV equation takes the form,

$$\frac{dx}{(1+x)(1+3x)} = -\frac{dy}{4(1-y)} \quad (16)$$

Integrating,

$$ln\left(\frac{1+x}{1+3x}\right) = -\frac{1}{2}ln(1-y) + k \quad (17)$$

Here k is the constant of integration. At the surface of the star, pressure is zero (boundary condition), so that, p=0 (x=0) at r=R

Let us take at r=R, $y = y_0 = \frac{2GM}{c^2 R}$ where, M is the mass of the star and R being the radius.

So, we get, $k = \frac{1}{2}ln(1-y_0)$

Now, the solution becomes on simplification,

$$x = \frac{\sqrt{1-y_0} - \sqrt{1-y}}{\sqrt{1-y} - 3\sqrt{1-y_0}} \quad (18)$$

At centre of the star, r=0, so, y=0, then let $x = x_0$ (boundary condition). Central pressure of the star can now be expressed as, $p_0 = \rho c^2 x_0$. Again, $x_0 = \frac{\sqrt{1-y_0}-1}{1-3\sqrt{1-y_0}}$

Therefore,

$$p_0 = \rho c^2\left[\frac{\sqrt{1-y_0} - 1}{1 - 3\sqrt{1-y_0}}\right] \quad (19)$$

Now, if the denominator of the above expression is zero, then central pressure goes to infinity and the star cannot maintain its stability, it will be gravitationally collapsed to a black hole. This scenario will occur when, $3\sqrt{1-y_0}=1$ or, $y_0=\frac{8}{9}$,

Hence, $\frac{2GM}{c^2\left(\frac{3M}{4\pi\rho}\right)^{\frac{1}{3}}}=\frac{8}{9}$

On simplification it follows [10]:

$$M=\left[\frac{4c^3}{\sqrt{243\rho\pi G^3}}\right] \quad (20)$$

Numerical Solution of the TOV Equation

To solve the **TOV equation** for stellar structure numerically, we implemented a Python script using the **Runge-Kutta method (RK45)**. The TOV equation models the relationship between the mass (m) and pressure (P) as a function of the radius (r). We used the scipy.integrate.solve_ivp() function, which applies as stepsize **RK45 solver**, ensuring accuracy and efficiency. The differential equations were solved over a radial range [10^{-5}, R] with an initial central mass m_0 and pressure $P_0 = 10^{-2}$. To prevent unphysical solutions, we introduced a **pressure threshold event** that stops the integration when the pressure drops below 10^{-10}, marking the surface of the star. The relationship between pressure P and density ρ is modeled using the polytropic equation of state:

$$P=K\rho^{\gamma}=K\rho^{\left(1+\frac{1}{n}\right)} \quad (21)$$

where, K is the Polytropic constant, determined by the star's initial conditions, Adiabatic index is γ, and n is the Polytropic index. n = 3 stands for relativistic degenerate matter, and n=1.5 for non-relativistic degenerate matter. For this study, we have used a polytropic index n = 3 or γ=4/3 [12] which is appropriate for relativistic stars composed of degenerate matter.

Finally, we plotted the **mass and pressure profiles** for different initial masses as 2.0, 4.0, and 8.0 solar mass, demonstrating how the stellar structure changes with varying central mass. The boundary conditions employed as, at the core (r = 0), m(0) = 0, P(0) = P_C (central pressure), and at the surface (r = R), P(R) = 0, where R is the star's radius.

Results

1. Mass and radius profile for different initial masses

The figure 1 illustrate the **Mass and Pressure profiles** for different initial masses m_0 as a function of the radius $R_{\odot}$. These visual aids are essential for clear understanding of the relation between mass, radius, and pressure distribution in the modeled system. The left panel of figure 1 shows the mass distribution as a function of the radial coordinate

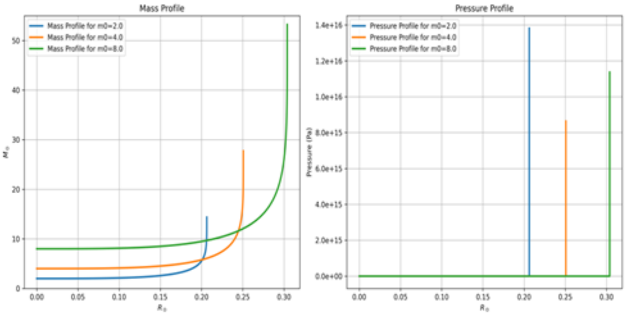

Figure 1: Mass and pressure profile for different initial masses using TOV and polytrope equation.

$R_{\odot}$revealing that higher mass cores exhibit steeper gradients near their surfaces, indicative of more concentrated mass distribution. The right panel shows the pressure profile for the same cores, demonstrating that as the initial mass increases, the central pressure grows significantly, which is a critical factor in determining the stability of the core. Mass radius and pressure radius profile of a star depends on its initial mass. The TOV equation (6) describes the pressure gradient inside the relativistic star, balancing pressure support and gravitational pull. This equation ensures hydrostatic equilibrium, where the outward pressure gradient counteracts the inward gravitational pull. The mass continuity equation (7) relates the mass distribution to the radial density profile.

Mass Profile: The mass profile exhibits the following findings:

Inner Core Uniformity: For small radii, the mass remains nearly constant which indicates a dense, uniform core structure across all initial mass configurations. For initial mass $2M_{\odot}$, mass remains constant up to $0.125R_{\odot}$, For $4M_{\odot}$ it remains constant up to $0.225R_{\odot}$, and for $8M_{\odot}$ it is up to $0.275R_{\odot}$. This uniformity is a consequence of the balance between gravitational and pressure forces within the dense core, where the polytropic equation of state enforces nearly constant density.

Outer Layers: Beyond the core region, the mass increases steeply, reflecting a rapid transition to less dense outer layers. The final masses and radii for the different configurations are:

For, the initial mass $2M_{\odot}$ final mass is $14.5M_{\odot}$ at $0.207R_{\odot}$

For, the initial mass $4M_{\odot}$ final mass is $27.7M_{\odot}$ at $0.251R_{\odot}$

For, the initial mass $8M_{\odot}$ final mass is $53.5M_{\odot}$ at $0.304R_{\odot}$

This steep behavior suggests that while the core holds a smaller fraction of the total mass, the outer regions of the star contribute significantly to the total mass. Furthermore, a linear relationship is observed between the final mass and the initial mass of the star. This proportionality suggests that for the given initial conditions and polytropic index, the total mass scales linearly with the initial core mass, though this relationship may vary with different polytropic indices or initial conditions.

Pressure Profile: The pressure profiles show marked transition:

Low-Pressure Core: For small radii, pressure is relatively low, supporting the uniform core structure.

High-Pressure Outer Layers: Pressure rises sharply beyond the core: For the initial mass $2M_{\odot}$ peak pressure is $1.3 \times 10^{16} Pa$ at radius $0.205R_{\odot}$, for the initial mass $4M_{\odot}$ peak pressure is $1.1 \times 10^{16} Pa$ at radius $0.250R_{\odot}$, and for the initial mass $8M_{\odot}$ peak pressure is $1.4 \times 10^{16} Pa$ at radius $0.305R_{\odot}$.

2. Solution of Hydrostaic equation and density profile

We investigate the mass limit and properties of neutron stars by solving the hydrostatic equilibrium equation using a customized density profile [11]. The degeneracy pressures for non-relativistic and relativistic regimes are incorporated into the analysis [10,11]. In the previous study, the mass-radius relation for neutron stars was investigated using the hydrostatic equilibrium (HE) equation in conjunction with a specific density profile. While this approach provided initial insights, it became evident that the HE equation alone fails to fully capture the relativistic effects essential for accurately modeling compact objects like neutron stars. For instance, the relation $MR^3 = 1.5 \times 10^{42}$, derived under simplified assumptions, predicts a decreasing mass with increasing radius as shown below (Figure 2).

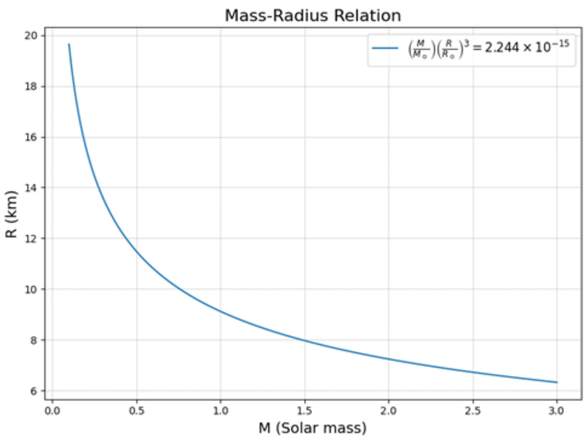

Figure 2: Mass Profile from Analytical Solution using equation of HE, Density Profile, and Degenerate Pressure

Analytical and numerical solution obtained as follows:

1. Analytical Calculation: Using degeneracy pressure for the core neutrons, the maximum mass of the neutron star was calculated analytically as $M_{\max} = 2.75M_{\odot}$ [11].

2. **Numerical Simulation:** The numerical integration of the mass continuity equation with the density profile as shown in Figure 2 rose monotonically with the radius, reaching $M = 2.84M_{\odot}$ for R=15Km. This result is slightly higher than the analytically predicted maximum mass.

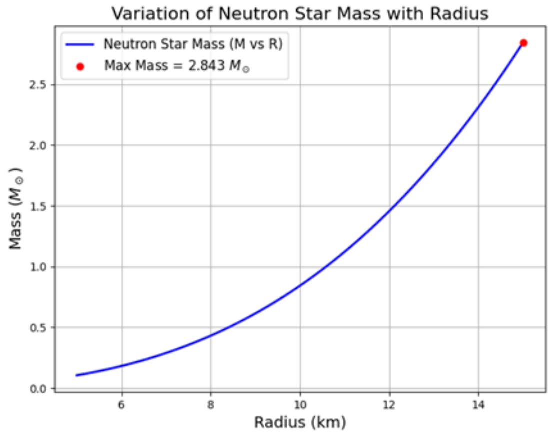

Figure 3: Mass Profile from Numerical Solution using equation of HE, Density Profile, and Degenerate Pressure.

3. TOV Equation and Density Profile

We have here Introduced the significance of neutron stars, emphasizing their unique structure governed by degeneracy pressure and gravitational forces. We have highlighted the existing mass limits (TOV limit) and the theoretical refinement using different density profiles. Starting from equation (20), we can write the relation as:

$$M = \frac{7.175 \times 10^{39}}{\sqrt{\rho}} \text{kg} = \frac{3.61 \times 10^{9}}{\sqrt{\rho}} M_{\odot} \tag{22}$$

Where, $M_{\odot}$ is the solar mass. The challenge here is selecting a suitable value for ρ , the density. Assuming the neutron star's density is analogous to that of a neutron, we estimate ρ using the relation $\rho = 2\frac{m_n}{\frac{4}{3}\pi r_n^3}$, where m_n kilogram is the mass of a neutron and $r_n \approx 10^{-15}$ meters is the neutron radius. This yields a neutron star density of approximately $8 \times 10^{17} \text{kg/m}^3$. Applying this density in the mass-density relation from equation (20), the calculated maximum mass of a neutron star is about $4.036M_{\odot}$, which notably exceeds experimental values.

4. Stability of neutron star dependence on density

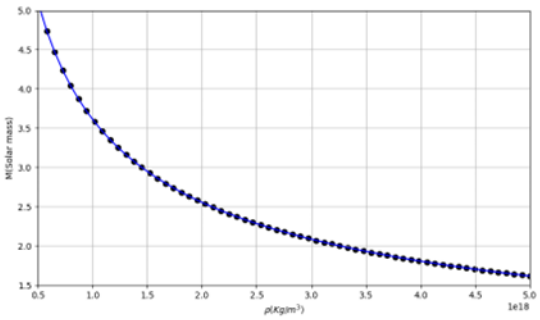

Figure 4: Dependence of Maximum Mass on Density using the TOV Solution.

Figure 4 illustrates the relationship between the maximum mass and density, based on this TOV solution. The plot shows that the limiting mass of neutron star to retain stability gets reduced with the increase of density of neutron star. At density greater than$5 \times 10^{18}\text{Kg/m}^3$ mass limit is less than $1.4M_{\odot}$ (Chandrasekhar mass limit of White Dwarf). This can not be the real scenario of existence of neutron star. So, neutron star might also possess a limiting density less than $5 \times 10^{18}\text{kg/m}^3$ to retain stability as obtained in this approximate solution of TOV equation considering density to be constant.

5. Mass limit of neutron star using TOV and density profile

Finally, we assume a central pressure of $1.2 \times 10^{34}\text{Pa}$ for the neutron star on the eve of collapsing, leading to an estimated maximum density of $1.6 \times 10^{18}\text{Kg/m}^3$. Substituting this density into equation (20) the resulting maximum mass for a neutron star is estimated approximately as $2.85M_{\odot}$, aligning more closely with the observational constraints. Further, by incorporating our predicted density profile (equation 8) into the TOV equation (equation 6) and running a Python simulation to plot limiting mass of neutron star vs its radius, we obtain a critical mass of $2.096M_{\odot}$ and a critical radius of 10.5 Km, with a maximum mass of $2.36M_{\odot}$ as shown in Figure 5.

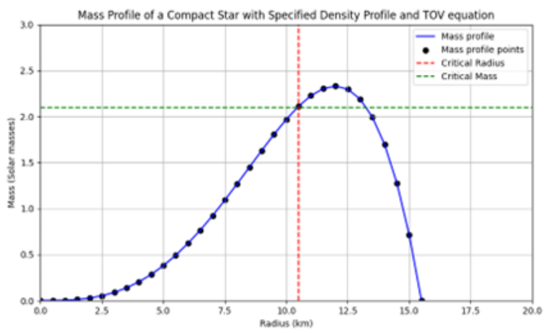

Figure 5: Mass Profile from Numerical Solution using TOV, Density Profile, and Degenerate Pressure.

Critical mass is defined as the limit for non-rotating neutron stars, beyond which neutron stars collapse into black holes, on the other hand maximum mass is the highest mass a neutron star that can be achieved before collapsing in black hole, potentially increased by factors like rotation. Figure 5 indicates an upper mass limit of neutron star, beyond which mass gets decreased to zero. This means that there might be an upper limit of radius of neutron star, no neutron star with greater radius than this upper limit can exist in nature which should be around 15 Km.

Discussion

1. Mass and Radius Profile for Different Initial Masses

The variation in peak pressure with initial stellar mass can be attributed to differences in density gradients and the relation between gravitational and pressure forces. The numerical results, based on the TOV equation and the polytropic equation of state, reveal the following key findings:

A **linear scaling** of the maximum mass with the initial core mass is observed, along with **transition zones** in both the mass and pressure profiles. These transition zones separate the uniform core from the steep outer gradients. The pressure profile shows distinct behavior with varying initial masses: When the initial mass increases from **2.0 to 4.0 solar masses**, the peak pressure **decreases**. However, when initial mass increases further to **8.0 solar masses**, the peak pressure **increases**. This discrepancy is likely due to **transition phenomena** within the stellar structure. For the 4.0 solar mass configuration, the density gradient creates a more gradual transition in density, resulting in a lower peak pressure. In contrast, at 8.0 solar masses, the higher overall density leads to an elevated peak pressure.

The influence of the **polytropic index (γ = 4/3)**, characteristic of relativistic degenerate matter, plays a critical role in the pressure-density relationship. This index significantly affects the pressure variance across different mass profiles. These mass and radius profiles illustrate how increasing central pressure and mass concentration destabilize the stellar core, eventually exceeding the critical thresholds for neutron star stability. They also provide comparative visualizations that indicate the conditions under which a collapsing core transitions into a black hole. Thus, this analysis offers valuable insights into the internal structure of stars and helps to determine critical mass thresholds, bridging the gap between theoretical models and observed gravitational collapse phenomena.

2. Solution of the Hydrostatic Equation and Density Profile

The observed discrepancy between the numerical simulation and the analytical results can be explained by several factors:

Simplified Density Profile: The numerical simulation uses a **prescribed density profile** without feedback from the pressure or a realistic equation of state (EoS), limiting its accuracy.

Lack of Instabilities: The non-relativistic framework does not account for **gravitational instabilities**, which are essential for predicting the maximum mass and its subsequent decrease.

Relativistic Effects: Analytical calculations include **relativistic effects** through the EoS for neutron degeneracy pressure. These effects govern the balance between gravity and degeneracy pressure, enabling the prediction of the maximum stable mass.

Limitations and Implications

The primary limitation of this approach is its inability to capture the **realistic mass decrease** beyond the maximum stable configuration observed in the analytical results. This shortcoming arises due to the absence of relativistic corrections to gravity and pressure. Additionally, feedback mechanisms between pressure and density are neglected, and the simplified density profile does not accurately reflect the complex internal structure of neutron stars. While the non-relativistic numerical simulation provides **qualitative insights** into the mass-radius relation, it fails to capture the maximum mass and its subsequent decrease caused by gravitational instability. However, the results remain consistent with observational constraints and offer valuable insights into the transition between **non-relativistic and relativistic regimes** in neutron star cores.

3. Mass Limit of Neutron Stars Using the TOV Equation and Density Profile

The results demonstrate that solving the **mass-continuity equation** numerically with the same density profile yields an increasing mass with radius, reflecting the realistic accumulation of matter within the star. However, this trend exposes the limitations of the **Hydrostatic Equilibrium (HE)** framework in high-density regimes.
The discrepancy arises because the HE equation neglects **relativistic effects**, such as spacetime curvature and relativistic pressure gradients, which become significant in dense astrophysical systems. To overcome these limitations, we incorporated the **TOV equation**, which extends the HE framework by including general relativistic effects.

By applying the TOV equation to the same density profile: The inconsistencies observed with the equation were resolved. The results aligned more closely with **realistic neutron star physics**. The corrected mass-radius relation provided deeper insights into the interrelation between gravitational and pressure forces within neutron stars.

Neutron Star Mass Constraints

The observed maximum mass of neutron stars is indeed close to **2.08** solar masses, with one of the most massive confirmed examples being **PSR J0740+6620**, measured at approximately **2.08–2.17** solar masses [13]. Some models suggest that the true upper limit for neutron star masses may lie between **2 and 3 solar masses**. The discrepancy between different models arises from the **uncertainty in the equation of state** for neutron stars. Despite significant progress in modeling, the exact EoS remains uncertain, as the innermost core of neutron stars is still not fully understood.

Conclusion

This study presents a comprehensive numerical analysis of the **Tolman-Oppenheimer-Volkoff (TOV) equations**, coupled with various density profiles, to investigate the critical mass and structural properties of neutron stars. The findings highlight the intricate relationship between **mass, pressure, and radius** in determining the stability of relativistic stars. Specifically, the results reveal significant **non-linearities in the mass and pressure distributions**, suggesting the potential existence of a dense outer shell within the stellar structure. These features may indicate underlying physical phenomena, such as **phase transitions** or variations in the **equation of state (EoS)** at high densities. Throughout this study, different methods were employed to estimate the **maximum mass limit** of a neutron star before it undergoes gravitational collapse into a black hole or another compact object. The key conclusions derived from the analyses are summarized as follows:

1. Equation of Hydrostatic Equilibrium: Analytical calculations of the **hydrostatic equilibrium (HE)** equation, combined with a customized density profile, yielded a maximum neutron star mass of approximately **2.75 solar masses**. Numerical simulations, based on integrating the **mass-continuity equation** with the same density profile, produced a consistent maximum mass estimate of **2.84 solar masses**.

2. Constant Density Model: By solving the TOV equations with a **constant density as** $1.6 \times 10^{18} \mathrm{Kg/m^3}$, the maximum mass was determined to be approximately **2.85 solar masses**. This result aligns closely with the theoretical predictions by **Hartle and Sabbadini (1977)**, demonstrating the reliability of the TOV framework when applied to simple density assumptions.

3. **Customized Density Profile:** Introducing the predicted density profile into the TOV equations yielded a critical mass of approximately **2.09 solar masses** at a radius of around **10 km**. The maximum mass was calculated as **2.36 solar masses** with the stellar mass density tapering to zero at a radius of **15.5 km**. This customized approach provided a refined understanding of the EoS for neutron stars, enabling more precise estimates of **critical mass thresholds**.

4. Implications and Consistency with Observations: Based on these findings, we conclude that a neutron star is likely to **lose stability and collapse** into a black hole or another compact object once it surpasses a mass of approximately **2.09 solar masses**. This conclusion is consistent with empirical findings, such as the upper mass limit established by **researchers at Goethe University Frankfurt** (January 16, 2018), who determined that the maximum mass of a neutron star cannot exceed **2.16 solar masses**.

While the possibility of extremely massive neutron stars beyond **2.3 solar masses** cannot be entirely ruled out [14], the results presented here align well with current **experimental and observational constraints**, providing a robust approximation of the maximum mass limit for neutron stars.

The estimated **2.09 solar mass** limit is consistent with both recent observational data and current theoretical models. Observations of the most massive neutron stars, such as: **PSR J0952-0607** ($2.35 M_\odot$) [15] and **PSR J0348+0432** ($2.01 M_\odot$) [16] demonstrate that neutron stars can indeed reach or exceed this mass threshold.

Additionally, theoretical models indicate that the maximum gravitational mass for **non-rotating neutron stars** is approximately $\mathbf{2.25 \pm 0.07} M_\odot$, further supporting the plausibility of the limit obtained in this study.

5. Population Studies and EoS Constraints: Population studies reveal that while most neutron stars cluster around $\mathbf{1.35–1.50} \boldsymbol{M_\odot}$, those in **high-mass X-ray binaries** or **millisecond pulsars** tend to be more massive, often exceeding $\mathbf{2.0} \boldsymbol{M_\odot}$. The mass limit of $\mathbf{2.09} \boldsymbol{M_\odot}$ fits well within this observational range, aligning with the most massive confirmed neutron stars.

The maximum neutron star mass also serves as a crucial **observational constraint** on the EoS of dense matter. Recent measurements of neutron stars with masses close to $\mathbf{2.0} M_\odot$ have ruled out many softer EoS models, suggesting that the maximum mass likely lies between **2.2 and 2.9**$M_\odot$ (Özel & Freire, 2016) [17]. Given the observational and theoretical constraints, a maximum neutron star mass of approximately **2.09 solar masses** emerges as a reasonable and well-supported upper limit. This value is consistent with current astrophysical observations and aligns with established theoretical models, lending **credibility to its scientific validity**. The conclusion is further reinforced by peer-reviewed references, affirming that **2.09 solar masses** represent a justifiable estimate for the maximum mass of neutron stars.

Implications and Future Work

These findings concentrate on the need for more sophisticated theoretical models that account for the influences of gravitational, nuclear, and quantum forces under relativistic conditions on such compact stars. Future works will focus on additional complexities to refine these models further. For instance, the inclusion of phase transitions in dense anisotropic pressure models, and the effects of strong magnetic matter (e.g., quark deconfinement or hyperon formation), fields may significantly alter predictions of neutron star structure and stability. Such enhancements could bridge the gap between theoretical predictions and astrophysical observations. Gravitational wave signals from neutron star mergers, combined with pulsar timing measurements and X-ray observations of thermal emissions, offer complementary avenues for testing and validating the models proposed in this study. The unavailability of a definitive and authentic EOS remains a significant challenge, our results highlight the power of theoretical modelling and numerical simulations in probing the unveiling physics of neutron stars.

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